Asset pricing dynamics in complete markets

Dawna Jones *

Department of Mathematics, Florida State University

July 2, 2013

Abstract

The standard general equilibrium asset pricing models typically undertake two common assumptions of homogeneous agents and rational expectations equilibrium. However, this context sometimes yields outcomes that are inconsistent with reality like negligible trading volume. In order to explain the overwhelming evidence of trading volume, I would need to develop a model where the usual no-trade theorems fail to hold. If the agents are perfectly rational then it would prove difficult to implement a model that violates the no-trade theorems. Therefore, I have sought to implement an artificial asset market where the agents are, instead, boundedly rational, utility maximizing, infinitely lived and forward looking. I restrict each agent's abilities by allowing them to use only information which is currently available. In this artificial asset market, I allow multiple risky assets and one risk-free bond. The economy starts out with the agents being endowed with a portion of each risky asset. At each time period, agents are faced with two choices: consumption choice and investment allocation. For each of these choices, an agent needs particular functions which only depend on current holdings and current prices. I require that these functions, when provided with equilibrium prices, give the same consumption and investment choices as that of the congruent general equilibrium model. Nevertheless, when agents are out of equilibrium, they will simultaneously solve the aforementioned functions along with predictive pricing functions at each time period. Also, the agents will be endowed with adaptive learning schemes to learn the true motions of the pricing functions.

This asset pricing model is being implemented with independent and identically distributed asset returns, agents are able to compute their policy functions using Hakansson's model [5]. In this framework, there are a few interesting questions. What are the market dynamics when agents are out of equilibrium? What are the particular plausible

*djones3@math.fsu.edu

out of equilibrium behaviours that leads to the agents learning the equilibrium? What is the nature of the evolution of the market clearing asset prices?

This paper is an expository piece on a boundedly rational asset pricing model. In order to develop an understanding of the context for this model, I will take sometime to explain the well-known Lucas asset pricing model [6]. In this model there is a single representative agent standing in for a number of homogeneous agents. This representative agent seeks to maximize his expected discounted lifetime utility by making the best possible decisions between consumption and investment choices.

$$\mathbb{E}\left\{\sum_{t=0}^{\infty}\beta^t U(c_t)\right\} \tag{1}$$

In Lucas' model, there is one consumption good, c_t, which is magically produced from n distinct units which I will, hereafter, refer to as stocks. Outputs from these stocks, $y_t = \{y_{it}|i = 1, \dots, n\}$, are perishable so that at each period the aggregate consumption is the sum of the outputs from all the stocks $c_t = \sum_{i=1}^{n} y_{it}$. Ownership of these stocks are competitively determined on a stock exchange. Each stock features one perfectly divisible equity share which follows a Markov process, $\mathbb{P}\left(y_{t+1} = y'|y_t = y\right)$. An equity share gives the owner of the stock at the beginning of a particular time period all the output of the stock which can be sold for p_{it}. In this representative agent economy, there will be no trading. All output is consumed and all shares are held. The interesting issue becomes the equilibrium asset price behaviour. According to Lucas [6], there is only one way for this economy to be in a competitive equilibrium, that is, all asset shares are held and the asset prices follow:

$$p_{it} = \mathbb{E}\left\{\beta\frac{U(c_{t+1})}{U(c_t)}(p_{i(t+1)} + y_{i(t+1)})|y_{it}\right\} \tag{2}$$

Agents are assumed to know a whole lot about the structure of the economy and perform complex computations. In a nutshell, this is a rational expectations model.

Rational expectations models embodies two components [8]. First, an agent is fully rational so much so that his decisions are as a result of optimizing an objective function subject to some constraints, Second, the constraints that every agent face must be mutually consistent. That is, the decisions of an agent also form constraints on other agents. It is as if the agents have to collude in order to make good choices. There are some draw backs to having an economy populated with fully rational agents. For example, if all agents in the theoretical economy are fully rational as in Muth's rational expectations theory [7], then there will be no trade. Suppose, there is a trader with insider information who wants to sell an asset in the market.

The other traders would immediately rationalize that he has negative information about that asset and will sought to sell the asset as well. There will not be anyone in the market willing to take the buy side of that asset. This is in stark contrast to what occurs in real markets, which is characterized by large trading volume on a daily basis. Also, an economy endowed with fully rational agents will never experience speculative bubbles. Because, as soon as the price of an asset seem out of sync with its value, these impossibly rational agent will immediately jump to exploit this arbitrage opportunity. These behaviours will keep all mispricings in check. However, in reality speculative bubbles persist for much longer than a moment. Moreover, such an economy would also not experience price volatility.

For me, It becomes natural to wonder at the nature of the pricing dynamic if the agents were a little less rational. That is, instead of assuming that the agents know everything about the economy and each other, the agent has access only to publicly and observable information with which he develops systematically imperfect forecasting rules. He would later adjusts his forecasting rule as more information are revealed with time. This is not a new idea. Guan [4] carried out an experiment by reducing the level of intelligence of the agents in the Lucas [6] framework. She [4] decided that the assumptions of homogeneous agents and rational expectations were too restrictive and sought to relax them by introducing a correct expectations equilibrium with boundedly rational heterogeneous agents.

A correct expectations framework consists of an aggregate pricing function and policy functions for each agent such that he optimizes his objective function with respect to publicly available information and personal preferences. Also, all markets should clear. The main difference between the fully rational agent and the boundedly rational agent is the amount of information that is used to make decisions. That is, agents' wealth, risk preferences and policy functions are available information on the market level but not on the individual agent's level. The fully rational agent will have access to all information on the market level, whereas the boundely rational agent will not.

Guan [4] used a discretized, infinite horizon market with N agents and one stock paying dividends each period. The ex-dividend price of the stock was denoted p_t and the dividend d_t. An agent is allowed to hold a positive fraction of the stock s_t^i during any given period and his wealth would be $w_t^i = s_t^i(p_t + d_t)$. Each period, an agent has to decide his consumption and his investment allocations under his budget constraints:

$$c_t^i + s_{t+1}^i p_t = s_t^i(p_t + d_t). \tag{3}$$

Also, the stock market should clear: $\sum_{i=1}^N s_t^i = 1$ which implies that $\sum_{i=1}^N c_t^i = d_t$. Guan simplified her model by allowing the dividend to be determined by an exogenous independent and identically distributed process

z_t:

$$\frac{d_{t+1}}{d_t} = z_{t+1} \tag{4}$$

where $d_0 = 1$ and each agent knows his initial stock holding.

At the beginning of each period, the dividend is announced. Each agent knows his own stock holding but not the market price. At this point, each agent would determine an optimal investment policy function function $s^i_{t+1} = s^i_t(p_t)$ which is a monotone decreasing function of the stock price. Next, all agents submit their policy functions to a clearing house which would use these functions to announce a price that would clear the stock market. Subsequently, the agents would trade into a new stock position and consume the positive remainder of their wealth, $c^i_t = w^i_t - s^i_{t+1}p_t$.

This whole process rests solely on how the agents determine their optimal investment and policy functions. As it turns out, it is a very delicate balance when the agent has limited information. In Guan's setup, the agents observe the dividends and knows the probability distribution of the dividend shocks, z_t. However, they do not know the holdings or preferences of the other agents. And, they do not know the probability distribution of the stock price. Nevertheless, these agents were endowed with learning rules to help them learn the motion of the stock price. Guan found out that in many cases at least one agent dies before the system stabilizes. This could be due to the absence of a risk-free asset and the limited number of risky assets. Or, it could be because her market was incomplete. However, in very special cases the system stabilizes and an equilibrium is found.

I want to create a similar market except add a risk-free asset and the option of more dividend paying stocks. Also, I believe it would be much nicer if the market was complete. Again, the big question would be the determination of the policy functions for each agent. For this problem, I have used the Hakansson's [5] model as an inspiration for the agents' policy funtions. The result is similar, however, the proof is a little more straight forward that the one in [5].

1 The Model

This is a model of an individual's economic decision under risk. Bear in mind that the agent do not know the distribution of the asset prices. They will, at the implementation of the model be required to produce an educated guess. For now, I will proceed as if the agent's guess is correct at equilibrium. A discrete time dynamic programming approach is taken so that the portfolio composition decision, the financing decision and the consumption decision are analysed simultaneously. The following notation and assumptions will be employed.

c_t : amount of consumption in period j, where $c_t \geq 0$

$U(c_1, c_2, c_3, \ldots)$: the utility function, defined over all possible consumption programs.
The class of functions to be considered is that of the form

$$U(c_1, c_2, c_3, \ldots) = u(c_1) + \beta U(c_2, c_3, c_4, \ldots) \tag{5}$$

$$= \sum_{j=1}^{\infty} \beta^{j-1} u(c_j), \quad 0 < \beta < 1. \tag{6}$$

It is assumed that $u(c)$ is monotone increasing, twice differentiable, and strictly concave for $c \geq 0$. The objective in each case is to maximize $\mathbb{E}[U(c_1, c_2, c_3, \ldots)]$, that is, the expected utility derived from consumption over all time.
w_t : amount of capital on hand at the beginning of time t.
M : the number of available assets, one of which is a risk-free asset which can be sold or bought.
S : number of available states.
α_i : the proportion of investment wealth invested in the asset i.
δ : the proportion of wealth that is consumed each period.
z_{it} : the amount of wealth invested in asset i in period t.
$R_f - 1$: the rate of interest on the risk-free asset.
R_i : the gross return on asset i. The distribution is assumed to be iid and have the following properties:

$$R_M = R_f \tag{7}$$

$$0 \leq R_i < \infty, \qquad (i = 1, 2, \ldots, M-1) \tag{8}$$

$$\mathbb{P}\left\{ \sum_{i=1}^{M-1} (R_i - R_f)\alpha_i < 0 \right\} > 0 \tag{9}$$

for all finite α_i and $\alpha_j \neq 0$ for at least one j.
Condition (9) can be referred to as the "no-easy-money condition". In essence this condition states:

1. that no combination of productive investment opportunities exists which provides with probability 1 a return at least as high as the borrowing rate of interest,

2. that no combination of short sale exists in which the probability is zero that a loss will exceed the lending rate of interest, and

3. that no combination of productive investments made from the proceeds of any short sale can guarantee against any loss.

That is, (9) may be viewed as a condition that the prices of the various asset must satisfy in equilibrium.

1.1 Assumptions:

1. Consumption and investment decisions are made at the beginning of each period.
2. The amount allocated to consumption is assumed to be spent immediately or, if spent gradually over the period, to be set aside in a nonearning account.
3. Any debt incurred by the individual must at all times be fully secured. That is, his debt cannot exceed the present value of his endowments at the end of any period.

1.2 Formulating the Problem for an Agent:

The relation which determines the amount of capital on hand at each time in terms of the amount on hand at the previous time is:

$$w_{t+1} = R_f z_{M,t} + \sum_{i=1}^{M-1} R_i z_{i,t} \tag{10}$$

where

$$\sum_{i=1}^{M} z_{i,t} = w_t - c_t \tag{11}$$

The first term on the right-hand-side of (10) represents the proceeds from savings, and the second term the proceeds from productive investments. Combining (10) and (11) gives,

$$w_{t+1} = \sum_{i=1}^{M-1} (R_i - R_f) z_{i,t} + R_f (w_t - c_t) \tag{12}$$

This is the difference equation which governs the wealth process.

Consider now the value function $V_t(w_t)$ which is the expected utility obtainable from consumption over all future time, evaluated at time t, when capital at that point is w_t. Assuming that the agent is able to accurately and intelligently guess the distribution of the asset returns, an optimal strategy ensue. The formal definition of the value function, $V_t(w_t)$ is:

$$V_t(w_t) = \max \mathbb{E}\left[U(c_t, c_{t+1}, c_{t+2}, \ldots)|w_t\right] \tag{13}$$

By the time-separability of the utility function, the value function may be rewritten as

$$V_t(w_t) = \max \mathbb{E}\left[u(c_t) + \beta \left\{\max \mathbb{E}\left[U(c_{t+1}, c_{t+2}, \ldots)|w_{t+1}\right]\right\}|w_t\right]. \tag{14}$$

Because, $\{R_i\}$ are assumed to be independent and identically distributed with respect to time, I have

$$V_t(w_t) = \max u(c_t) + \beta \mathbb{E}\left[V_{t+1}(w_{t+1})|w_t\right] \tag{15}$$

Also, I would be faced with the same problem at time $t+1$ as at time t, and the time subscript may be dropped. Then,

$$V(w) = \max_{c,\{z_i\}} u(c) + \beta \mathbb{E}\left[V(\sum_{i=1}^{M-1}(R_i - R_f)z_i + R_f(w-c))|w\right] \tag{16}$$

The problem that each individual faces each period is:

$$V(w) = \max_{c,z_i} u(c) + \beta \mathbb{E}\left[V(w')|w\right] \tag{17}$$

subject to:

$$\begin{aligned} c &\geq 0 && (18)\\ w' &= R_f z_M + \sum_{i=1}^{M-1} R_i z_i && (19)\\ \sum_{i=1}^{M} z_i &= w - c && (20)\\ \mathbb{P}\left\{\sum_{i=1}^{M-1}(R_i - R_f)\, z_i + R_f(w-c) \geq 0\right\} &= 1 && (21) \end{aligned}$$

1.3 Simple case solution:

To make a sketch of the solution, I consider an artificial financial market where $M = 2$. Also, according to Duffie [3], $c_t = \delta w_t$, $0 < \delta \leq 1$ when the portfolio return is independent and identically distributed. The problem (17) can be paraphrased in terms of proportions instead of 'dollar' amounts:

$$V(w) = \max_{\delta, \alpha} u(\delta w) + \beta \mathbb{E}\left[V(w')|w\right] \tag{22}$$

subject to:

$$\begin{aligned} 0 < \quad \delta &< 1 && (23)\\ w' &= w(1-\delta)\left(\alpha R + (1-\alpha)R_f\right) && (24)\\ \mathbb{P}\left\{(R - R_f)\,\alpha + R_f \geq 0\right\} &= 1 && (25) \end{aligned}$$

For this discrete model, (25) means that $(R - R_f)\,\alpha + R_f \geq 0$ is true for each state.

To find the first order conditions, differentiate (22) with respect to the choice variables and equate to zero:

$$u'(\delta w) = \beta \mathbb{E}\left[V'(w')\left(\alpha R + (1-\alpha)R_f\right)|w\right] \tag{26}$$

$$\beta \mathbb{E}\left[V'(w')\left(R - R_f\right)|w\right] = 0 \tag{27}$$

To find the envelope condition, differentiate problem (22) with respect to the state variable, w, and simplify to get

$$V'(w) = u'(\delta w) \tag{28}$$

Then, update (28) one period and substitute into equations (26) and (27) to get the Euler equations.

$$u'(\delta w) = \beta \mathbb{E}\left[u'(\delta w')\left(\alpha R + (1-\alpha)R_f\right)|w\right] \tag{29}$$

$$\beta \mathbb{E}\left[u'(\delta w')\left(R - R_f\right)|w\right] = 0 \tag{30}$$

With some help from Cochrane [2], I can rewrite (30) as:

$$\beta \mathbb{E}\left[u'(\delta w')R|w\right] = \beta \mathbb{E}\left[u'(\delta w')R_f|w\right] = u(\delta w) \tag{31}$$

In order to simplify the Euler equations, I specify that the agent has constant relative risk aversion. His utility function is $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$. Therefore, (29) becomes:

$$(\delta w)^{-\gamma} = \beta \mathbb{E}\left[(\delta w')^{-\gamma}\left(\alpha R + (1-\alpha)R_f\right)|w\right] \tag{32}$$

Use (24) to simplify:

$$\frac{(1-\delta)^{\gamma}}{\beta(1-\gamma)} = \mathbb{E}\left[u\left(\alpha R + (1-\alpha)R_f\right)|w\right], \quad \gamma \neq 1. \tag{33}$$

So, for $\gamma \neq 1$, let

$$K = \max_{\alpha} \mathbb{E}\left[u\left(\alpha R + (1-\alpha)R_f\right)|w\right] \tag{34}$$

and

$$\delta = 1 - \left(K\beta(1-\gamma)\right)^{\frac{1}{\gamma}}. \tag{35}$$

On the other hand, when $\gamma = 1$ then $\delta = 1 - \beta$. In summary, the agent's decision rules are given below

$$
\begin{aligned}
z &= \alpha(1-\delta)w && (36)\\
z_f &= (1-\alpha)(1-\delta)w && (37)\\
c &= \delta w && (38)\\
\delta &= 1-(K\beta(1-\gamma))^{\frac{1}{\gamma}}, \quad \gamma \neq 1 && (39)\\
\delta &= 1-\beta, \quad \gamma = 1 && (40)\\
K &= \max_{\alpha} \mathbb{E}\left[u\left(\alpha R+(1-\alpha)R_f\right)|w\right] && (41)
\end{aligned}
$$

It is interesting to note that an agent does not need to know anything about the other agents in the economy to implement his decision rules. The only missing piece to the puzzle, from the perspective of the agent, is the distribution of the asset prices.

2 Future Work

Now that I have the decision rules for the boundedly rational agents, I can move on to test if they are satisfactory for implementation. I do this by developing an equivalent but relatively simple rational equilibrium model, solve for the price distribution and then give this to the boundedly rational agent. If asset markets and the consumption market clears, then I would have a winner. Doing this is not as simple as it sounds though. Because, solving a general equilibrium model with heterogeneous agents is not quite as easy. However, by using Benninga and Mayshar's result [1]:

$$\gamma_R = \frac{1}{\sum_{i=1}^{N} \frac{\eta_i}{\gamma_i}} \tag{42}$$

which offers a way to calculate the representative agent's constant relative risk aversion rate γ_R in terms of the individual agents' risk aversion rates, γ_i, and negishi weights, η_i. Roughly speaking, the negishi weight of an agent measures how much influence that agent has on the economy. Benninga and Mayshar [1] also offers a way to computationally obtain the negishi weights. However, to get a somewhat decent result for an infinite time model, a finite time model is implemented with a large final time. The implementation is very slow and does not yield a satisfactory level of accuracy. Presently, I only have one decimal place of accuracy. I am working on improving this.

Subsequently, I will need to develop a learning scheme for each agent so as to endow them with a means by which to learn the correct asset price distribution. Next, I will seek to implement the model to answer the following questions. What are the market dynamics when agents are out of equilibrium? What are the particular plausible out of equilibrium behaviours that leads to the agents learning the equilibrium? What is the nature of the evolution of the market clearing asset prices?

References

[1] Simon Benninga and Joram Mayshar. Dynamic wealth redistribution, trade and asset pricing. March 1993.

[2] John Cochrane. *Asset Pricing*. Princeton University Press, Princeton, New Jersey, revised edition, 2005.

[3] Darrell Duffie. *Security Markets: Stochastic Models*. Academic Press, San Diego, California, 3rd edition, 1988.

[4] Yuanying Guan. *Asset Market Dynamics of Heterogeneous Agent Models with Learning*. PhD thesis, Florida State University, 2011.

[5] Nils Hakansson. Optimal investment and consumption strategies under risk for a class of utility functions. *Econometrica*, 38:587–607, September 1970.

[6] Robert Lucas. Asset prices in an exchange economy. *Econometrica*, 46(6):1429–1445, November 1978.

[7] John Muth. Rational expectations and the theory of price movements. *Econometrica*, 29(3):315–335, July 1961.

[8] Thomas Sargent. *Bounded Rationality in Macroeconomics*. Oxford University Press, New York, 1993.